

Bertotti-Robinson and Melvin Spacetimes

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Similarities between the Melvin and Bertotti-Robinson spacetimes are discussed and a uniqueness conjecture is formulated.

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I. INTRODUCTION

The Bertotti-Robinson (BR) and Melvin (ML) spacetimes have interesting similarities. Both are static, axisymmetric solutions of the Einstein-Maxwell equations. Each solution has one free parameter which characterizes the strength of the electromagnetic field. In this paper we will concentrate on one particular property: that both spacetimes are static Einstein-Maxwell solutions that are geodesically complete. To see why this is striking, consider the analogous case of static, geodesically complete solutions of the vacuum Einstein equation. A theorem of Lichnerowicz [1] states: The only geodesically complete stationary vacuum spacetime (M,g) which is asymptotically flat is Minkowski spacetime (R^4,η) . Anderson [2, 3] has considerably strengthened this result by allowing one to deduce asymptotic flatness, not require it. He has generalized Lichnerowicz's theorem as follows:

Theorem (Anderson). *The only geodesically complete stationary vacuum spacetime (M,g) is Minkowski spacetime (R^4,η) .*

To prove this generalization, Anderson makes use of the geometric properties of the Ernst equations, satisfied by a stationary vacuum spacetime. Since static electrovac spacetimes also satisfy Ernst equations, one might expect that the results of Anderson could be generalized to the electrovac case. However, Anderson also notes that the analog of his theorem for Einstein-Maxwell solutions is false because the Melvin solution is a counterexample. Furthermore, the method of proof [2, 3] cannot be generalized to the electrovac case because the electrovac Ernst equations have different geometric properties than the vacuum Ernst equations.

There remains the question of whether static, geodesically complete electrovac solutions are numerous or rare. It is our opinion that such solutions are rare. We make the following conjecture.

CONJECTURE: *The only geodesically complete static Einstein-Maxwell spacetimes are Melvin and Bertotti-Robinson*

We have not been able to prove this conjecture. However, the remainder of this paper will be a plausibility argument for the conjecture. For simplicity, we will confine ourselves to electrovac spacetimes that are axisymmetric as well as static. In section II we will consider the vacuum case (i.e. the well known Weyl solutions) and examine which properties of this class of solutions prevent the non-flat ones from being geodesically complete. In section III we consider the Einstein-Maxwell equations and examine the properties of the ML and BR spacetimes that allow them to be geodesically complete. Section IV has further discussion, and details of the ML and BR spacetimes are contained in Appendices.

II. VACUUM SOLUTIONS

A vacuum or electrovac static, axisymmetric metric can be written in the Weyl-Levi-Civita (WLC) form:

$$ds^2 = -e^{2U} dt^2 + e^{-2U} [e^{2K} (d\rho^2 + dz^2) + \rho^2 d\varphi^2] \quad (1)$$

where U and K are functions of ρ and z . In the vacuum case, U satisfies the flat space Laplace equation [4]

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} = 0 \quad (2)$$

while K is determined by U through the following equations:

$$\frac{\partial K}{\partial \rho} = \rho \left[\left(\frac{\partial U}{\partial \rho} \right)^2 - \left(\frac{\partial U}{\partial z} \right)^2 \right], \quad (3)$$

$$\frac{\partial K}{\partial z} = 2\rho \frac{\partial U}{\partial \rho} \frac{\partial U}{\partial z}. \quad (4)$$

The solutions of Eq.(2) can be written in closed form. They are linear combinations of singular solutions of the form

$$U = r^{-(\ell+1)} P_\ell(\cos \theta) \quad (5)$$

and non-singular solutions of the form

$$U = r^\ell P_\ell(\cos \theta). \quad (6)$$

Here r and θ are given by $r = \sqrt{\rho^2 + z^2}$ and $\theta = \tan^{-1}(\rho/z)$. P_ℓ denotes the Legendre polynomial of order ℓ . It is not surprising that the singular U of Eq.(5) gives rise to a singular spacetime. But why does the non-singular U of Eq.(6) also give rise to a singular spacetime? At first one might suspect that perhaps the metric function K is singular, but it is easy to see that that is not the case. The U solutions of Eq.(6) are polynomials in ρ and z from which it follows, using Eqs.(3) and (4), that K is also a polynomial in ρ and z . Instead, the answer is found by examining curvature invariants, in particular the Kretschmann scalar. For a static vacuum spacetime we have

$$C^{abcd}C_{abcd} = 8E^{ab}E_{ab} \quad (7)$$

Here E_{ab} is the electric part of the Weyl tensor defined by

$$E_{ac} = C_{abcd}n^b n^d \quad (8)$$

where n^a is the unit vector in the direction of the static timelike Killing field. From Eq.(1) and the fact that E_{ab} is trace-free, it then follows that

$$C^{abcd}C_{abcd} = 4e^{4(U-K)} [(E_{\rho\rho} - E_{zz})^2 + 3(E_{\rho\rho} + E_{zz})^2 + 4(E_{\rho z})^2]. \quad (9)$$

The electric part of the Weyl tensor can be expressed in terms of U by

$$E_{ab} = D_a D_b U + D_a U D_b U \quad (10)$$

where D_a is the derivative operator associated with the spatial part of the metric. From Eq.(9) it is clear that an unbounded Kretschmann scalar can be caused by the quantity $U - K$ being unbounded above. Thus, in the WLC metrics that come from the nonsingular U of equation (6) the curvature blows up “at infinity” (*i.e.* at large ρ and z) because $e^{4(U-K)}$ blows up at infinity. Strictly speaking, a blowup of curvature “at infinity” does not make a spacetime singular unless geodesics can get “to infinity” in a finite affine parameter. However, since spatial distance in the ρ or z direction is determined by the quantity e^{K-U} it is not surprising that when this quantity goes to zero at infinity some geodesic can get there in finite affine parameter.

III. EINSTEIN-MAXWELL SOLUTIONS

For magnetostatic axisymmetric solutions of the Einstein-Maxwell equations, there is a metric of the WLC form of Eq.(1) together with a Maxwell field. The field F_{ab} has zero electric component, $F_{ab}n^a = 0$, with the magnetic component determined from a scalar potential ψ :

$$F_{tb}^* = \partial_b \psi \quad (11)$$

where F_{ab}^* is the dual of F_{ab} . The metric function U and the scalar potential ψ satisfy the following equations [4]:

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} = e^{-2U} \left[\left(\frac{\partial \psi}{\partial \rho} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 \right], \quad (12)$$

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 2 \frac{\partial U}{\partial \rho} \frac{\partial \psi}{\partial \rho} + 2 \frac{\partial U}{\partial z} \frac{\partial \psi}{\partial z}. \quad (13)$$

The metric function K is determined by U and ψ from the following equations:

$$\frac{\partial K}{\partial \rho} = \rho \left[\left(\frac{\partial U}{\partial \rho} \right)^2 - \left(\frac{\partial U}{\partial z} \right)^2 \right] + \rho e^{-2U} \left[\left(\frac{\partial \psi}{\partial z} \right)^2 - \left(\frac{\partial \psi}{\partial \rho} \right)^2 \right], \quad (14)$$

$$\frac{\partial K}{\partial z} = 2\rho \frac{\partial U}{\partial \rho} \frac{\partial U}{\partial z} - 2\rho e^{-2U} \frac{\partial \psi}{\partial \rho} \frac{\partial \psi}{\partial z}. \quad (15)$$

The Melvin solution has functions

$$U = \ln \left(1 + \frac{1}{4} B_0^2 \rho^2 \right) \quad (16)$$

$$\psi = B_0 z \quad (17)$$

$$K = 2 \ln \left(1 + \frac{1}{4} B_0^2 \rho^2 \right) \quad (18)$$

where B_0 is a constant representing the strength of the magnetic field.

At large ρ we have $U \approx 2 \ln \rho$. However $U = 2 \ln \rho$ is actually a vacuum solution, one of the well known Levi-Civita solutions. For this solution $K = 4 \ln \rho$ and thus $e^{U-K} \rightarrow 0$ as $\rho \rightarrow \infty$. It then follows from Eq.(9) that the Kretschmann scalar also goes to zero as $\rho \rightarrow \infty$. However, unlike the ML solution, the vacuum Levi-Civita solution is singular on the axis. Thus, the ML solution seems to be a delicate compromise: at infinity it approaches a non-flat vacuum solution, so U must blow up at infinity. However, unlike the polynomial solutions of Eq.(5) in the ML solution, U blows up sufficiently slowly at infinity so that the curvature

does not blow up there. The presence of the Maxwell field does not modify the behavior at infinity, but only serves to make this behavior compatible with non-singular behavior on the axis. It is the apparent delicacy of this compromise that leads to our conjecture.

We now turn to the Bertotti-Robinson solution to see whether it exhibits similar behavior. For this solution we have

$$U = \ln \lambda + \frac{1}{2} \ln (\rho^2 + z^2) \quad (19)$$

$$\psi = \lambda (\rho^2 + z^2)^{1/2} \quad (20)$$

$$K = 1 \quad (21)$$

where λ is a constant setting the magnetic field strength. At fixed z and large ρ we have $U \rightarrow \ln \rho$. This is similar behavior to what is seen in the ML case, though for BR the Maxwell field remains constant and the solution does not approach any vacuum solution. Nonetheless, the appearance of $\ln \rho$ behavior is striking: once again a singularity at infinity is avoided by U increasing at a rate smaller than polynomial. The fact that both the ML and BR solutions do this, and that some property like this seems to be needed to avoid singular spacetime behavior at infinity, leads us to believe that precisely this logarithmic behavior is necessary.

IV. DISCUSSION

We have presented a uniqueness conjecture for the ML and BR spacetimes, and a plausibility argument for that conjecture. However, a plausibility argument is not a proof. It may be that the methods of Anderson [2, 3] could be modified in some way to provide such a proof. On the other hand, it may be that the conjecture could be falsified by finding a counterexample. The strongest part of our plausibility argument is for the logarithmic behavior of solutions at infinity where $U_{\text{ML}} \rightarrow \ln \rho^2$ and $U_{\text{BR}} \rightarrow \ln \rho$; but possibly other solutions share this behavior. One could search for such solutions numerically by using the methods of Headrick et al [10].

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Appendix A: Melvin Spacetime

The ML metric [5] describes a static magnetic field which is a bundle of magnetic flux lines in magnetostatic-gravitational equilibrium. Its line element is given by

$$ds^2 = a_B^2[-dt^2 + d\rho^2 + dz^2] + a_B^{-2}\rho^2 d\varphi^2 \quad (\text{A1})$$

with $a_B = 1 + B_0^2\rho^2/4$. The ML manifold has four Killing vectors $\partial_t, \partial_z, \partial_\varphi$ and $z\partial_t + t\partial_z$. These Killing vectors are respectively time translation, spatial translation along the axis, rotation around the axis, and boost along the axis.

The Maxwell field is given by

$$F = B_0\rho a_B^{-2} d\rho \wedge d\varphi \quad (\text{A2})$$

where B_0 is the value of the magnetic field on the $\rho = 0$ axis. As with any Einstein-Maxwell solution there are additional solutions given by duality rotation: that is, the metric is unchanged but the Maxwell tensor F_{ab} maps to $F_{ab} \cos \beta + F_{ab}^* \sin \beta$ where β is any constant and F_{ab}^* is the dual of F_{ab} .

The ML metric as given in Eq.(A1) is already in the WLC form of Eq.(1). Thus, we can immediately read off that

$$U = \ln a_B \quad (\text{A3})$$

$$K = \ln a_B^2 \quad (\text{A4})$$

The Maxwell invariants are

$$I_1 = \frac{1}{2}F_{ab}F^{ab} = B_0^2 a_B^{-4} \quad (\text{A5})$$

$$I_2 = \frac{1}{2}F_{ab}^*F^{ab} = 0 \quad (\text{A6})$$

The ML metric is Petrov type D with the only non-zero Weyl tensor component

$$\Psi_2 = -B_0^2 a_B^{-4} (1 - a_B/2). \quad (\text{A7})$$

The Kretschmann scalar is

$$R_{abcd}R^{abcd} = 4B_0^4 a_B^{-8} [2 + 3(1 - B_0^2\rho^2/4)^2] \quad (\text{A8})$$

while a similar scalar involving the Ricci tensor is

$$R_{ab}R^{ab} = 4B_0^4 a_B^{-8} \quad (\text{A9})$$

Note that both of these scalars vanish as $\rho \rightarrow \infty$.

Melvin and Wallingford [11] have computed all the geodesics paths in the ML spacetime, and it follows from their computation that ML is geodesically complete.

Appendix B: Bertotti-Robinson spacetime

The BR spacetime [6–9] has line element

$$ds^2 = \left(\frac{1}{\lambda^2 r^2}\right)(-d\tau^2 + dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2) \quad (\text{B1})$$

where λ is a constant. This spacetime is thus the direct product of the two-sphere and two dimensional anti de-Sitter spacetime. It therefore inherits the symmetries of both these spaces and has a six parameter isometry group. Since the line element is conformally flat, it follows that the Weyl tensor is zero. Furthermore, the large isometry group insures that all curvature scalars are constants.

Metric (B1) can be put in WLC form by the coordinate transformation

$$\tau = \lambda^2 t, \quad \frac{\sin \vartheta}{r} = \rho, \quad \frac{\cos \vartheta}{r} = z \quad (\text{B2})$$

which yields the line element

$$ds^2 = -\lambda^2(\rho^2 + z^2)dt^2 + \frac{1}{\lambda^2(\rho^2 + z^2)}(d\rho^2 + dz^2 + \rho^2 d\varphi^2) \quad (\text{B3})$$

Comparing (B3) with metric (1) provides

$$U = \ln \lambda + \frac{1}{2} \ln(\rho^2 + z^2) \quad (\text{B4})$$

$$K = 1 \quad (\text{B5})$$

The Maxwell field is given by

$$F = \frac{\rho}{\lambda(\rho^2 + z^2)^{3/2}}(z d\rho - \rho dz) \wedge d\varphi \quad (\text{B6})$$

The Maxwell invariants are

$$I_1 = \frac{1}{2} F_{ab} F^{ab} = \lambda^2 \quad (\text{B7})$$

$$I_2 = \frac{1}{2} F_{ab}^* F^{ab} = 0 \quad (\text{B8})$$

The Kretschmann scalar is

$$R_{abcd} R^{abcd} = 8\lambda^4 \quad (\text{B9})$$

while a similar scalar involving the Ricci tensor is

$$R_{ab} R^{ab} = 4\lambda^4 \quad (\text{B10})$$

The geometric structure of BR is $S^2 \otimes AdS_2$. Therefore (as pointed out *e.g.* by Clément and Gal'tsov [12]) since each of these two dimensional spaces is geodesically complete it follows that BR is geodesically complete.

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